Příklad: $f(x,y) = x + 2y + \frac{3}{4}x^2 + xy + 2y^2$

$M = \{(x,y)\in\mathbb{R}^2 : \ y^2-2 \le x \le 2-y^2\}$

$H(M) = \{ y^2-2 = x, \ y \in [-\sqrt{2},\sqrt{2}]\} \cup$
$\cup \{ 2-y^2 = x, \ y \in [-\sqrt{2},\sqrt{2}]\}.$

najdeme extrémy přes $H(M)$ pomocí V68.

$H_1 = \{(x,y)\in\mathbb{R}^2 : \ x = y^2-2\} =$
$= \{(x,y)\in\mathbb{R}^2 : \ \underbrace{y^2-x-2}_{g_1(x,y)} = 0\}$

$g_1(x,y)$ ... vazbová fce

(1 RCE)

Hledáme body $(\tilde{x},\tilde{y}) \in H_1$, že $\exists \lambda \in \mathbb{R}$

$$\nabla f(\tilde{x},\tilde{y}) + \lambda \nabla g_1(\tilde{x},\tilde{y}) = 0$$ (2 RCE)

$f_x(x,y) = 1 + \frac{3}{2}x + y$ | $g_{1x}(x,y) = -1$

$f_y(x,y) = 2 + x + 4y$ | $g_{1y}(x,y) = 2y$

SOUSTAVA ROVNIC:

$y^2 - x - 2 = 0$ (1)

$1 + \frac{3}{2}x + y + \lambda\cdot(-1) = 0$ (2)

$2 + x + 4y + \lambda\cdot 2y = 0$ (3)

(1): $x = y^2-2$ dosaď do (2,3):

$x = -2 \lor x = -1$

$1 + \frac{3}{2}(y^2-2) + y - \lambda = 0$ (2')

$2 + y^2 - 2 + 4y + 2\lambda y = 0$ (3')

dosaď (2') ($\lambda = \dots$) do (3'):

$\cancel{2} + y^2 \cancel{-2} + 4y + 2\left(1 + \frac{3}{2}(y^2-2) + y\right)y = 0$

$y^2 + (2 + 3y^2 - 6 + 2y)y + 4y = 0$

$y^2 + 2y + 3y^3 - 6y + 2y^2 + 4y = 0$

$3y^3 + 3y^2 - 4y + 4y = 0$ | $3y\cdot(y^2+y) = 0$
$3y^2(y+1)$

$y = 0 \lor y = -1$

PB.
$[-2,0]$,
$[-1,-1]$

$H_2 = \{(x,y)\in\mathbb{R}^2 : x = 2-y^2\} =$

$= \{(x,y)\in\mathbb{R}^2 : \underbrace{y^2+x-2}_{g_2(x,y)} = 0\}$

$(g_2)_x(x,y) = 1 \qquad (g_2)_y(x,y) = 2y$

$$\boxed{x = 2-y^2} \qquad (1)$$

$$1+\frac{3}{2}x + y + \lambda\cdot 1 = 0 \qquad (2)$$

$$2 + x + 4y + \lambda 2y = 0 \qquad (3)$$

$$1+\frac{3}{2}(2-y^2)+y+\lambda=0 \qquad (2')$$

$$2+2-y^2+4y+2\lambda y=0 \qquad (3')$$

$$4-\frac{3}{2}y^2 + y+\lambda = 0 \Rightarrow \lambda = \frac{3}{2}y^2 - y - 4$$

$$4 - y^2 + 4y + 2\lambda y = 0$$

$$4-y^2+4y+y\cdot(3y^2-2y-8)=0$$

$$4-\cancel{y^2}+4y+3y^3-\cancel{2y^2}-8y=0$$

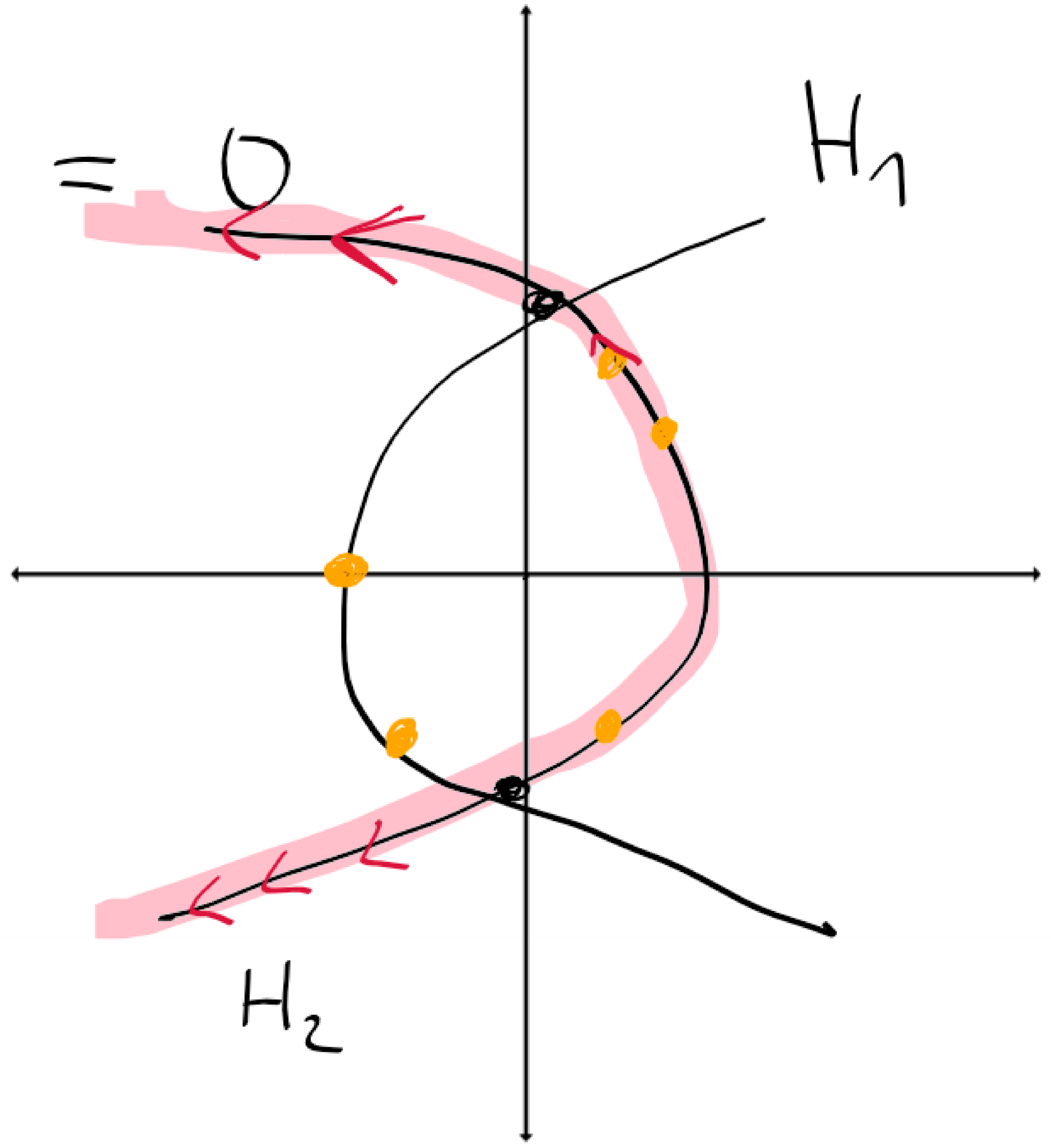
$$3y^3-3y^2-4y+4=0$$

$$3y^2(y-1)-4(y-1)=0$$

$$(y-1)(3y^2-4)=0$$

$y=1 \vee$ | $y=\pm\sqrt{\frac{4}{3}}$

$x=1$ | $x=\frac{2}{3}$



P.B.: $[1,1]$, $\left[\frac{2}{3}, \sqrt{\frac{4}{3}}\right]$, $\left[\frac{2}{3}, -\sqrt{\frac{4}{3}}\right]$.

Jsou tyto body (P.B.) $\in H(M)$?

(zjistíme, že ano.) | P.B. jsou ještě průsečíky obou parabol.

Extrémy fce $f(x,y) = 2x + 8y$ $\qquad \nabla f = (2,8)$

na množině $M = \{(x,y)\in\mathbb{R}^2 : x^2+2y^2=1\}$

$$= \{(x,y)\in\mathbb{R}^2 : \underbrace{x^2+2y^2-1}_{g(x,y)} = 0\}$$

$\nabla g(x,y) = (2x, 4y)$

3 ROVNICE:

- $g(x,y) = 0$
- $\frac{\partial f}{\partial x}(x,y) + \lambda \frac{\partial g}{\partial x}(x,y) = 0$
- $\frac{\partial f}{\partial y}(x,y) + \lambda \frac{\partial g}{\partial y}(x,y) = 0$

Tj: $\nabla f + \lambda \nabla g = 0$

POZNÁMKA body, v nichž $\nabla g = 0$ jsou také podezřelé z extrému.

(nesplňují předpoklad věty!)

M je omezená jasné. M je uzavřená:

$M = \underbrace{g}_{\text{spoj.}}{}^{-1}(\underbrace{\{0\}}_{\text{uz.}})$ je uzavřená.

M je uz. a om.

$\iff$ M je komp.

$\implies$ extrémy se nabývají

Příklad: $f(x,y,z) = x + 2y - z$

na $M = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 \le 1\}$

$M = \{x^2+y^2+z^2 = 1\}$ ... Lagr

$\cup \{x^2+y^2+z^2 < 1\}$ ... hledáme SB (lokální ext.)

$x^2 + 2y^2 = 1$

$2 + \lambda\cdot 2x = 0 \qquad x = -\frac{1}{\lambda} = \alpha \qquad -\frac{1}{\lambda} = \alpha$

$8 + \lambda\cdot 4y = 0 \qquad y = \frac{-2}{\lambda} = 2\alpha$

$\alpha^2 + 2(2\alpha)^2 = 1$

$9\alpha^2 = 1 \qquad \alpha = \pm\sqrt{\frac{1}{9}} \qquad -\frac{1}{\lambda} = \pm\sqrt{\frac{1}{9}}$

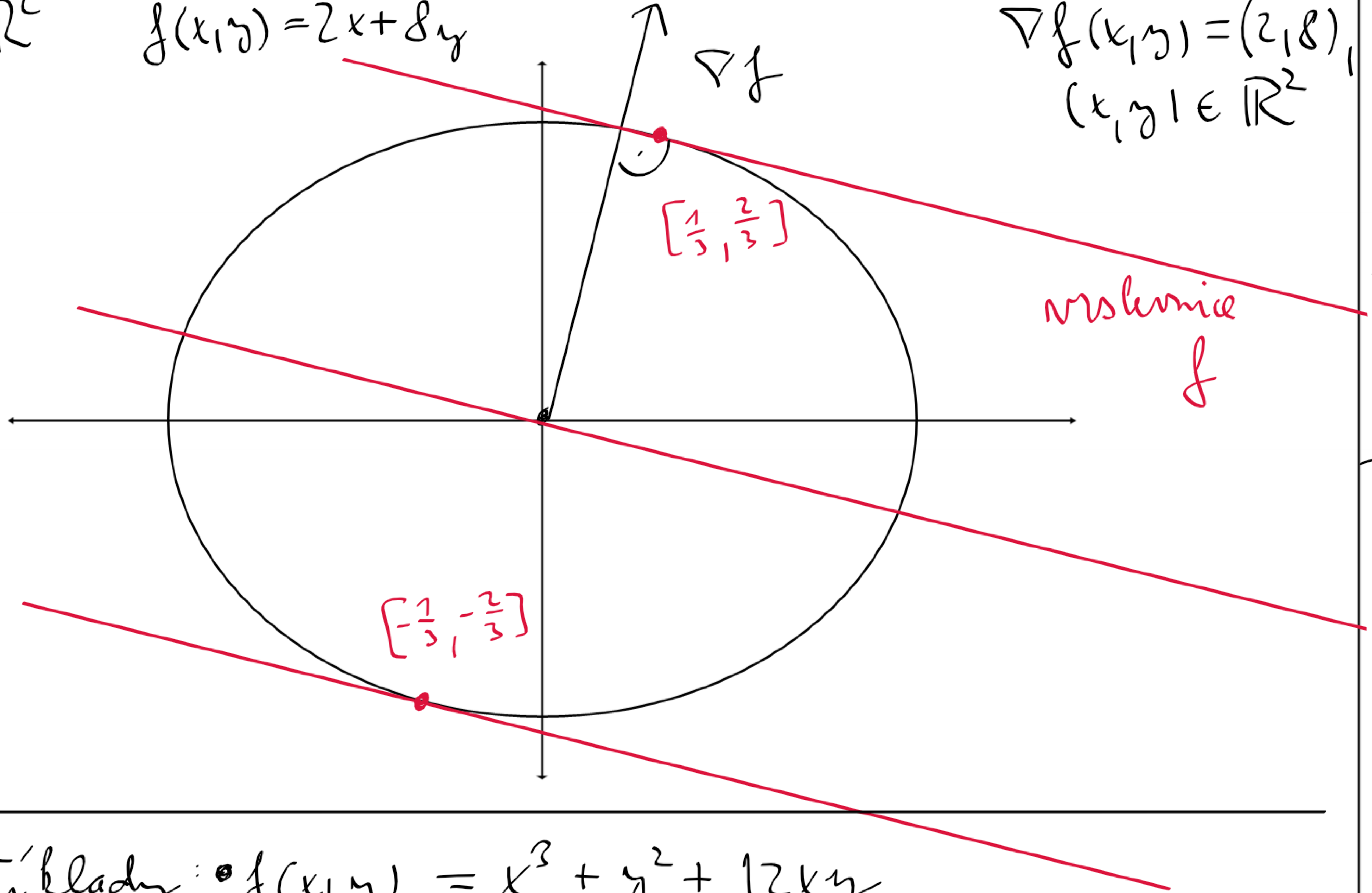
$\lambda = \pm\sqrt{9}$

$x = \frac{-1}{\sqrt{9}}, \; y = \frac{-2}{\sqrt{9}}$

$x = \frac{1}{\sqrt{9}}, \; y = \frac{2}{\sqrt{9}}$

P.B.: $[-\frac{1}{3}, \frac{-2}{3}], [\frac{1}{3}, \frac{2}{3}]$

$\mathbb{R}^2$ $\quad f(x,y) = 2x + 8y$ $\qquad \nabla f(x,y) = (2,8),\ (x,y) \in \mathbb{R}^2$



Příklady: • $f(x,y) = x^3 + y^2 + 12xy$

$[0,0]$ ... není lok. extrém

$[24,-144]$ ... lok. minimum

• $f(x,y,z) = x^3 + y^3 + z^3 - 3xy - 3xz - 3yz$

$[0,0,0]$ ... není lok. extrém

$[2,2,2]$ ... lok. minimum

• $f(x,y) = xy + \frac{50}{x} + \frac{20}{y}$, $\quad x>0, y>0$

$[0,0]$ – lok. min., $\quad [-\frac{1}{4}, -\frac{1}{2}]$ není lok. extr.

POSTUP: 1) Hledání SB, tj. bodů, kde

$$\frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0 \quad \left(\frac{\partial f}{\partial z} = 0\right)$$

2) V těchto PB určit matici $d^2 f(a)$ 2. PD. (Tj. matice 2×2, resp. 3×3)

určit její povahu (PD, ND, PSD, NSD, ID)

Př1: SB:

$f_x(x,y) = 3x^2 + 12y = 0 \Rightarrow 3x^2 - 72x = 0$

$f_y(x,y) = 2y + 12x = 0 \qquad y = -6x$

$x^2 - 24x = 0 \qquad x = 0 \Rightarrow y = 0$

$x(x-24) = 0 \qquad x = 24 \Rightarrow y = -144$

P.B. $[0,0]$, $[24,-144]$

$$f(x,y) = x^3 + y^2 + 12xy \qquad f_x = 3x^2 + 12y$$
$$f_y = 2y + 12x$$

$$d^2 f(x,y) = \begin{pmatrix} 6x & 12 \\ 12 & 2 \end{pmatrix}$$

$$d^2 f(0,0) = \begin{pmatrix} 0 & 12 \\ 12 & 2 \end{pmatrix} \sim \begin{pmatrix} 12 & 14 \\ 12 & 2 \end{pmatrix} \sim \begin{pmatrix} 26 & 14 \\ 14 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 26 & 14 \\ 0 & 2\cdot 26 - 14\cdot 14 \end{pmatrix} \sim \begin{pmatrix} \overset{>0}{26} & 0 \\ 0 & \overset{<0}{2\cdot 26 - 14\cdot 14} \end{pmatrix} \quad \text{ID mce}$$

$\Rightarrow$ $[0,0]$ není bod extrému.

$$d^2 f(24,-144) = \begin{pmatrix} \overset{>0}{144} & 12 \\ 12 & 2 \end{pmatrix} \Rightarrow \underline{\underline{PD}} \Rightarrow$$

$$\det = 144\cdot 2 - 12\cdot 12 = 144 > 0$$

$[24,-144]$ je bodem lok. minima

$$f(x,y,z) = x^3 + y^3 + z^3 - 3xy - 3xz - 3yz$$

$$f_x = \cancel{3}x^2 - \cancel{3}y - 3z = 0$$

$$f_y = \cancel{3}y^2 - \cancel{3}x - \cancel{3}z = 0$$

$$f_z = \cancel{3}z^2 - \cancel{3}x - \cancel{3}y = 0$$

$$y = x^2 - z$$

$$\Rightarrow$$

$$\Rightarrow z^2 - x - x^2 + z = 0$$

$$z^2 + z - (x^2 + x) = 0$$

$$z^2 + z = x^2 + x = y^2 + y$$

$$x^2 - y^2 = y - x$$

$$(x-y)(x+y) = y - x$$

$$(x-y)(x+y) = -(x-y)$$

pokud $x - y \neq 0$, dostaneme $x + y = -1$

$$x = -y - 1$$